Lab 1 Notes

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The given Pareto distribution is

$$f(x) = \frac{\alpha 3^\alpha}{x^{\alpha+1}}, \quad x > 3, \quad \alpha > 1,$$

and zero otherwise.

(i) Find the cdf.

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} \frac{\alpha 3^\alpha}{t^{\alpha+1}}\, dt \\
&= \int_{3}^{x} \alpha\, 3^\alpha\, t^{-(\alpha+1)}\, dt \\
&= \left. \frac{\alpha\, 3^\alpha\, t^{-(\alpha+1)+1}}{-(\alpha+1)+1} \right|_{t=3}^{t=x} \\
&= -3^\alpha\, t^{-\alpha} \Big|_{t=3}^{t=x} \\
&= -3^\alpha (x^{-\alpha} - 3^{-\alpha}) \\
&= -\frac{3^\alpha}{x^\alpha} + 1 \\
&= 1 - \left(\frac{3}{x}\right)^\alpha, \quad x > 3
\end{aligned}$$

and $F(x) = 0$ for $x \leq 3$.

(ii) Find the quantile function.

Recall that the cdf was defined as

$$F(x) = \mathbf{P}\left(X \leq x\right) = p,$$

where x is the *quantile* and is given, and p is the probability that we need to often need to calculate. With the quantile function, the conditions are reversed:

$$Q(p) = x \quad \text{such that} \quad \mathbf{P}\left(X \leq x\right) = p,$$

where p is given is x is the value we seek. If $F(x)$ is a one-to-one function, then $Q(p)$ is simply the inverse of $F(x)$.

Using the cdf from (i), we solve for x:

$$\begin{aligned} p &= 1 - \left(\frac{3}{x}\right)^{\alpha} \\ \left(\frac{3}{x}\right)^{\alpha} &= 1 - p \\ \frac{3}{x} &= (1-p)^{1/\alpha} \\ x &= \frac{3}{(1-p)^{1/\alpha}} \end{aligned}$$

Therefore, our quantile function is:

$$Q(p) = \frac{3}{(1-p)^{1/\alpha}}, \quad 0 < p < 1$$